

Harmonic Functions

The Laplacian, Harmonic Functions, and Harmonic Conjugates

Lecture Notes

September 22, 2026

1 From Analytic to Harmonic

We identify a point $(x, y) \in \mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$, and write $B(z_0, r)$ for the open disc of radius r about z_0 . Our aim is to isolate the single second-order equation satisfied by the real and imaginary parts of an analytic function.

Reminder — the classes C^2 and C^∞ . A real function g on an open set $\Omega \subseteq \mathbb{R}^2$ is of class C^k if all of its partial derivatives up to order k exist and are continuous on Ω . Thus $g \in C^2$ means that g and the derivatives $g_x, g_y, g_{xx}, g_{xy}, g_{yx}, g_{yy}$ are all continuous, while $g \in C^\infty$ means $g \in C^k$ for every k — all partial derivatives of every order exist and are continuous. For $g \in C^2$ the mixed partials agree, $g_{xy} = g_{yx}$ (Clairaut–Schwarz); this is the one calculus fact we lean on below.

Our starting point is a fact whose proof is deferred; it will follow later from the Cauchy integral formula.

Theorem (stated now, proved later)

Every function analytic on a disc $B(z_0, r)$ is infinitely complex differentiable there. In particular, its real and imaginary parts, viewed as functions of (x, y) , are of class C^∞ on $B(z_0, r)$.

The point we extract is that these parts satisfy a single second-order equation.

Corollary (real and imaginary parts)

Let $f = u + iv$ be analytic on $B(z_0, r)$, with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ real-valued. Then

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0 \quad \text{on } B(z_0, r).$$

Proof. By the theorem above $u, v \in C^\infty$, so all second partials exist and the mixed partials commute. Analyticity is equivalent to the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Differentiating the first with respect to x and the second with respect to y ,

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy},$$

and adding, using $v_{yx} = v_{xy}$,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

The computation for v is identical: differentiating the first equation with respect to y and the second with respect to x gives $v_{yy} = u_{xy}$ and $v_{xx} = -u_{yx}$, whence $v_{xx} + v_{yy} = 0$. $\square$

2 The Laplace Operator

The operator appearing on the left-hand side above deserves a name.

The Laplace operator

For $u \in C^2(\Omega)$ the **Laplace operator** (or **Laplacian**) is

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

Recall the **Wirtinger derivatives** from the previous section,

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

They factor the Laplacian.

Proposition (factorization of the Laplacian)

On $C^2(\Omega)$,

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$

Proof. Using the Wirtinger formulas,

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \cdot \frac{1}{2} (\partial_x - i \partial_y) \cdot \frac{1}{2} (\partial_x + i \partial_y) = (\partial_x - i \partial_y) (\partial_x + i \partial_y) = \partial_x^2 + \partial_y^2 + i(\partial_x \partial_y - \partial_y \partial_x).$$

On C^2 functions the mixed partials cancel, leaving $\partial_x^2 + \partial_y^2 = \Delta$; reversing the order of the factors gives $4 \partial_{\bar{z}} \partial_z$ in the same way. $\square$

This gives a one-line proof of the Corollary. If f is analytic then $\partial f / \partial \bar{z} = 0$ (the Cauchy–Riemann theorem from the previous lecture’s notes), so

$$\Delta f = 4 \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \bar{z}} \right) = 0;$$

and since Δ has real coefficients, $\Delta u = \operatorname{Re}(\Delta f) = 0$ and $\Delta v = \operatorname{Im}(\Delta f) = 0$.

3 Harmonic Functions: Definition and Examples

Definition (harmonic function)

Let $\Omega \subseteq \mathbb{C}$ be open. A function $u \in C^2(\Omega)$ is called **harmonic** on Ω if

$$\Delta u = 0 \quad \text{everywhere on } \Omega.$$

Unless stated otherwise u is real-valued; a complex-valued u is harmonic if and only if $\operatorname{Re} u$ and $\operatorname{Im} u$ are.

In this language the Corollary reads: *the real and imaginary parts of an analytic function are harmonic.* The examples below are precisely the real and imaginary parts of the simplest analytic functions.

- $u = x^2 - y^2 = \operatorname{Re}(z^2)$ **on** $\mathbb{C}$. Here $u_{xx} = 2$ and $u_{yy} = -2$, so $\Delta u = 0$.
- $u = xy = \frac{1}{2} \operatorname{Im}(z^2)$ **on** $\mathbb{C}$. Here $u_{xx} = u_{yy} = 0$, so $\Delta u = 0$ trivially.
- $u = e^x \cos y = \operatorname{Re}(e^z)$ **on** $\mathbb{C}$. We compute $u_{xx} = e^x \cos y$ and $u_{yy} = -e^x \cos y$, so $\Delta u = 0$.
- $u = \log |z| = \operatorname{Re}(\log z)$ **on** $\mathbb{C} \setminus \{0\}$. A direct computation gives $\Delta u = 0$ on the punctured plane. This example, harmonic away from the origin, will reappear later.

4 The Local Converse

The Corollary showed that the real part of an analytic function is harmonic. The converse holds, at least locally.

Theorem (local converse)

Let u be real-valued and harmonic on a disc $B(z_0, r)$. Then there exists a function f analytic on $B(z_0, r)$ such that

$$u = \operatorname{Re} f.$$

The function f is unique up to an additive imaginary constant.

The proof requires contour integration and is therefore postponed; once integrals are available we will establish a more general version, valid on any simply connected domain.

Definition (harmonic conjugate)

If u is harmonic on Ω , a real function v for which $u + iv$ is analytic on Ω is called a **harmonic conjugate** of u .